J. Fluid Mech. (1978), *vol.* 88, *part* 3, *pp.* 535–540
Printed in Great Britain

On the power laws for turbulent jets, wakes and shearing layers and their relationship to the principle of marginal instability

By MARTIN LESSEN

Department of Mechanical and Aerospace Sciences, University of Rochester, New York 14627†

(Received 22 July 1977 and in revised form 5 May 1978)

The classical power laws describing the similarity solutions for turbulent jets, wakes and shearing layers are found to determine a fixed turbulent Reynolds number for each flow. The power laws are then derived from the principle of marginal instability without the usual assumptions.

1. Introduction

Similarity solutions for jets, wakes and shearing layers have been known for many years and the respective power laws describing the variation of the characteristic width and average velocity scale along with the turbulent viscosity as a function of downstream distance are common to a number of models of turbulent transport. The power laws may be obtained in the manner of Reichardt (1941) and Goertler (1942) and summarized as in Schlichting (1968, p. 686).

The concept of a 'mixing length' analogous to a particle mean free path was first introduced by Prandtl (1925) in order to provide a foundation for a theory of turbulent mixing. Prandtl further assumed that, in a similarity turbulent flow, the mixing length would be proportional to the characteristic width scale and that the Eulerian time rate of change of the width scale would be proportional to the average transverse velocity. It is significant that the mixing-length model with associated assumptions and alternative transport theories yielded power laws which compared with the observed behaviour of similarity flow regions of actual turbulent jets, wakes and shearing layers to good approximation.

The power laws summarized in table 1 are interesting in that the Reynolds numbers of the various flows based on the respective characteristic width and velocity scales and the turbulent or eddy viscosities are all constant for each flow over the entire similarity flow fields. It has already been noted by Townsend (1956, p. 128) and Corrsin (1957) that the 'effective' Reynolds number (based on the eddy viscosity) is independent of the 'true' Reynolds number (based on the molecular viscosity), and in addition Corrsin observed that the 'effective' Reynolds numbers for the round jet and the plane wake were of the same order of magnitude as the 'lower critical Reynolds numbers' of laminar free shear layers. It should be noted that at the time of that observation no lower critical Reynolds numbers for the shear layer or the round jet had as yet been calculated.

† Temporary address: U.S. Office of Naval Research, 223/231 Old Marylebone Road, London NW1 5TH.

Flow	Width scale δ	Average velocity scale U_m	Turbulent viscosity ν_t
Free jet boundary	x	x^0	x
Two-dimensional jet	x	$x^{-\frac{1}{2}}$	$x^{\frac{1}{2}}$
Circular jet	x	x^{-1}	x^0
Two-dimensional wake	$x^{\frac{1}{2}}$	$x^{-\frac{1}{2}}$	x^0
Circular wake	$x^{\frac{1}{3}}$	$x^{-\frac{2}{3}}$	$x^{-\frac{1}{3}}$

TABLE 1. Power laws describing the variation of the characteristic width scale δ, the average velocity scale U_m and the turbulent (eddy) viscosity ν_t with downstream distance x.

The distinguishing feature of turbulent jets, wakes and shearing layers is the existence of a point of inflexion in the velocity profile, for the case of plane flows, or in the case of three-dimensional jets and wakes, the existence of a point in the velocity profile that behaves mathematically like a point of inflexion; such flows are known to exhibit strong inviscid instabilities and generally rather low minimum critical Reynolds numbers. The interaction of a disturbance with the aforementioned velocity distributions will generally not greatly affect the stability characteristics of the flows, in distinction to the case where the velocity profile has no point of inflexion. Malkus (1956) postulated an extremum principle for turbulent flows involving a wave-like disturbance of the mean flow similar to a Tollmien–Schlichting wave, but application to plane Poiseuille flow did not reproduce observations. However, it is to be noted that plane Poiseuille flow is not inviscidly unstable.

Since, in the case of turbulent jets, wakes and shearing layers, the average flow characteristics are constant in time, there must be a mechanism which maintains the turbulence and hence the eddy viscosity. Such a mechanism must transfer energy from the average (steady) motion to the random (?) or turbulent motion. It is hereby postulated that the mechanism consists of a normal-mode oscillation or Tollmien–Schlichting-like wave on the 'average' flow which then interacts with the full spectrum of the turbulence present to pump energy into it, the wave drawing its energy from the average motion. The question may then be asked, 'Is the wave damped, amplified or neutral?' If the wave were damped, a decreasing amount of energy would flow into the turbulence, resulting in a reduced eddy viscosity and a progressively higher Reynolds number based on the eddy viscosity; the process would continue until the *minimum critical Reynolds number for instability* was reached and then stop. If the disturbance were amplified, the reverse would happen, leading to the conclusion that the disturbance must be *neutral or marginally unstable with a dimensionless wave-number corresponding to that at the minimum critical Reynolds number*.

According to the principle of marginal instability, the flow Reynolds number (based on the eddy viscosity) should be equal to the minimum critical Reynolds number for instability. Since the turbulent flows considered have similarity solutions, the 'minimum critical' Reynolds number corresponding to each solution is fixed; therefore all similarity turbulent jet, wake and shearing-layer flows have correspondingly fixed turbulent Reynolds numbers.

The far-field flow of a turbulent jet was studied by Lessen & Singh (1974) on this basis and the experimentally observed angle of spread correlated very well with that predicted by the principle of marginal instability. The experiments on free turbulent shearing layers by Brown & Roshko (1974) are also in agreement with the principle; not only is the angle of spread fixed, but so is the dominant dimensionless wavenumber.

One might consider the area of 'inviscid' separated flows and jets as generally characterized by surfaces of velocity discontinuity. Since the surfaces of velocity discontinuity are unstable with respect to a travelling-wave disturbance, a 'mixing' zone will develop to replace the surfaces of discontinuity and spread at a characteristic angle with a characteristic dimensionless wavenumber. In the case of a plane jet or wake (top-hat velocity distribution), the near-field shearing mixing layers will spread to obliterate the core flow and will then continue to spread at the rate of spread corresponding to the far flow field. In the corresponding round jets and wakes, the rotationally symmetric mode of disturbance becomes stable as the core is obliterated but the modes of azimuthal periodicity unity take over with their own characteristic minimum critical Reynolds numbers, dimensionless wavenumbers and similarity spreading relationships. The point at which the angle of spread changes from its near-field to its far-field value is called the break point as in Lessen & Paillet (1976), where the predicted transition from the near to the far field according to the principle of marginal instability was shown to correspond to observations by Mattingly & Chang (1974).

The two-dimensional wake behind a plate or a bluff body exhibits an antisymmetric (kink) mode of instability at a lower critical Reynolds number than the symmetric (sausage) mode and hence it generally dominates the situation after the core flow is obliterated by the spreading of turbulent shearing layers. The dominant dimensionless wavenumber then corresponds to that of the von Kármán trailing vortex street.

The foregoing discussion leads to the conclusion that, even in 'inviscid' flows, surfaces of velocity discontinuity, though mathematically possible, are not physically realizable and that turbulent shearing layers spreading in a predictable way correspond to reality. It is therefore seen that the principle of marginal instability is in agreement with the known power laws describing the variation of width, velocity and turbulent viscosity with downstream distance in that it predicts a constant turbulent Reynolds number for each flow. It will now be demonstrated that the power laws themselves may be derived from this principle.

2. Analysis

Jets, wakes and shearing layers are generally considered to be 'almost parallel' flows. In the modelling of these flows, the boundary-layer approximation to the full Navier–Stokes equations is used. In addition, for the case of wake flow, the wake deficit velocity is considered to be much smaller than the free-stream velocity.

Two-dimensional turbulent shearing layers, jets and wakes

The relevant approximation to the Navier–Stokes equations may be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu_t\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where x and y are the co-ordinates parallel and normal to the principal flow directions, respectively, and u and v are the respective velocity components. $\nu_t = \nu_t(x)$ is the turbulent viscosity. For the case of turbulent shearing layers and jets, let

$$\chi = \int_0^x \nu_t(x^1)\,dx^1, \quad \eta = y\chi^p, \quad \psi = U_m \chi^n g(\eta),$$

where ψ is a stream function and $U_m = U_m(x)$ is a characteristic velocity scale. Since

$$u = \partial\psi/\partial y = U_m \chi^{n+p} g'(\eta),$$

it follows that $n = -p$. Therefore (2) becomes

$$\frac{dU_m}{dx} g'^2 + \left(p\nu_t U_m \chi^{-1} - \frac{dU_m}{dx}\right) g g'' = \nu_t \chi^{2p} g'''. \tag{3}$$

For the shearing layer, U_m = constant; therefore for (3) to yield a similarity solution, it is necessary that $p = -\frac{1}{2}$. Hence, from the definition of η, $\delta \sim \chi^{\frac{1}{2}}$. From the principle of marginal instability, the turbulent Reynolds number $R_t = U_m \delta/\nu_t$ = constant. Therefore $\nu_t = d\chi/dx \sim \chi^{\frac{1}{2}}$ and $\chi \sim x^2$. Finally, $\nu_t \sim x$ and $\delta \sim x$, in agreement with the classical power laws in table 1.

For the two-dimensional jet, it is necessary that $U_m \sim \chi^{2p+1}$ for (3) to yield a similarity solution. Also, from the principle of marginal instability, $\nu_t \sim \chi^{p+1}$ since $\delta \sim \chi^{-p}$. Therefore $x \sim \chi^{-p}$ and $\delta \sim x$. From momentum-flux conservation, $U_m^2 \delta$ = constant. Hence $U_m \sim x^{-\frac{1}{2}}$ and $\nu_t \sim x^{\frac{1}{2}}$ in agreement with table 1.

For the two-dimensional wake, (2) is further modified to give

$$U_\infty \partial u/\partial x = \nu_t \partial^2 u/\partial y^2, \tag{4}$$

where U_∞ is the free-stream velocity past the obstacle producing the wake. Let

$$\chi = \int_0^x \nu_t(x^1)\,dx^1, \quad \eta = y\chi^p, \quad \psi = U_\infty g(\eta).$$

It follows that $u = U_\infty \chi^p g' = U_m g'$ and (4) becomes

$$pU_\infty \chi^{p-1}(g' + g'') = \chi^{3p} g'''. \tag{5}$$

For a similarity solution, $p = -\frac{1}{2}$ and $\delta \sim \chi^{\frac{1}{2}}$, thus $U_m \sim \chi^{-\frac{1}{2}}$. From R_t = constant, ν_t = constant, therefore $\chi \sim x$, $\delta \sim x^{\frac{1}{2}}$ and $U_m \sim \chi^{-\frac{1}{2}}$ in agreement with table 1.

Three-dimensional turbulent jets and wakes

The relevant approximation to the Navier–Stokes equations is

$$\left.\begin{aligned} \frac{\partial(ru)}{\partial x} + \frac{\partial(ru)}{\partial r} &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} &= \frac{\nu_t}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right). \end{aligned}\right\} \tag{6}$$

For the case of the jet, let

$$\chi = \int_0^x \nu_t(x^1)\,dx^1, \quad \eta = r\chi^p, \quad \psi = U_m \chi^n g(\eta);$$

then
$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{U_m}{\eta} g' \chi^{n+2p},$$

from which $n = -2p$. Equation (6) can then be written as

$$\frac{g'}{\eta}\left(U_m \nu_t g'' + \left[(n+p)\, U_m \nu_t + \frac{dU_m}{dx}\chi\right]\frac{g'}{\eta}\right) - \left(\frac{g''}{\eta} - \frac{g'}{\eta^2}\right)$$
$$\times \left[pU_m \nu_t g' + \left(nU_m \nu_t + \frac{dU_m}{dx}\chi\right)\frac{g}{\eta}\right] = \nu_t \chi^{1-n}\left(\frac{g'''}{\eta} - \frac{g''}{\eta^2} + \frac{g'}{\eta^3}\right). \quad (7)$$

For a similarity solution $U_m \sim \chi^{1-n}$. Since $\delta \sim \chi^{-p}$ and R_t = constant,

$$\nu_t = d\chi/dx \sim \chi^{1-n-p} \quad \text{and} \quad x \sim \chi^{p+n}.$$

From conservation of momentum flux, $U_m^2 \delta^2$ = constant, or $U \sim \delta^{-1}$. Hence $p = 1-n$. Since $n = -2p$ as well, $p = -1$ and $n = 2$. Therefore $x \sim \chi$, ν_t = constant, $U_m \sim \chi^{-1}$ and $\delta \sim x$ in agreement with table 1.

For the case of the three-dimensional wake, (6) is further modified to

$$U_\infty \frac{\partial u}{\partial x} = \frac{\nu_t}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad (8)$$

which upon using

$$\chi = \int_0^x \nu_t(x^1)\, dx^1, \quad \eta = r\chi^p, \quad \psi = U_\infty g(\eta), \quad U = U_\infty \chi^{2p}\frac{g'}{\eta} = U_m \frac{g'}{\eta}$$

becomes

$$pU_\infty \chi^{-1}\left(\frac{g'}{\eta} + g''\right) = \chi^{2p}\left(\frac{g'''}{\eta} - \frac{g''}{\eta^2} + \frac{g'}{\eta^3}\right). \quad (9)$$

For a similarity solution, $p = -1$, $\delta \sim \chi^{\frac{1}{2}}$ and $U_m \sim \chi^{-1}$. Since R_t = constant, $\nu_t \sim \chi^{-\frac{1}{2}}$. Therefore $x \sim \chi^{\frac{3}{2}}$, $U_m \sim x^{-\frac{2}{3}}$, $\delta \sim x^{\frac{1}{3}}$ and $\nu_t \sim x^{-\frac{1}{3}}$ in agreement with table 1.

The solutions for the average velocity distributions of the similarity flows can be obtained from (3), (5), (7) and (9), respectively, in the usual manner.

3. Conclusions

The principle of marginal instability allows the derivation of the power laws describing the variation of the characteristic width and average velocity scales and the turbulent viscosity with downstream distance without any additional assumptions regarding the nature of the turbulence. The turbulent Reynolds numbers fix the variation rates from first principles without requiring evaluation of a parameter from experimental data. Furthermore, the wavenumber of the marginally unstable disturbance is related to the dominant eddy scale.

REFERENCES

BROWN, G. L. & ROSHKO, A. 1974 On density effects and large structure in turbulent mixing layers. *J. Fluid Mech.* **64**, 775.

CORRSIN, S. 1957 Some current problems in turbulent shear flows. In *Naval Hydrodynamics*, chap. 15. Nat. Acad. Sci. – Nat. Res. Counc. publ. 515.

GOERTLER, H. 1942 Berechnung von Aufgaben der freien Turbulenz auf Grund eines neuen Näherungsansatzes. *Z. angew. Math. Mech.* **22**, 244.

LESSEN, M. & PAILLET, F. L. 1976 Marginal instability of turbulent shearing layers and the break point of a jet. *Phys. Fluids* **19**, 942.

LESSEN, M. & SINGH, P. J. 1974 Stability of turbulent jets and wakes. *Phys. Fluids* **17**, 1329.

MALKUS, W. V. R. 1956 Outline of a theory of turbulent shear flow. *J. Fluid Mech.* **1**, 521.

MATTINGLY, G. E. & CHANG, C. C. 1974 Unstable waves in an axisymmetric jet column. *J. Fluid Mech.* **64**, 541.

PRANDTL, L. 1925 Über die ausgebildete Turbulenz. *Z. angew. Math. Mech.* **5**, 136.

REICHARDT, H. 1941 Über eine neue Theorie der freien Turbulenz. *Z. angew. Math. Mech.* **21**, 257.

SCHLICHTING, H. 1968 *Boundary-Layer Theory*, 6th edn. McGraw-Hill.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow*. Cambridge University Press.